Assignment 4

Supplementary Problems

Hand in: Supplementary Problems no. 2, 3, 5, 6, 7a.
Deadline: Feb 13, 2018.

1. Determine which of the following functions are convex/strictly convex:
   (a) $f_1(x) = x^p,\ x \in (0, \infty)$.
   (b) $f_2(x) = x^x\ , x \in (0, \infty)$.
   (c) $f_3(x) = \tan x\ , x \in (-\pi/2, \pi/2)$.
   (d) $f_4(x) = x \log x,\ x \in (0, \infty)$.
   (e) $f_5(x) = (1 + \sqrt{x})^{-1}, \quad x \in (-1, \infty)$.
2. Let f and g be two convex functions defined on I. Show that the function $h(x) = \max\{f(x), g(x)\}$ is convex. Is the function $j(x) = \min\{f(x), g(x)\}$ convex?
3. Give an example to show that the product of two strictly convex functions may not be convex. How about the composite of two strictly convex functions?
4. Let f be a convex function on (a, b) whose inverse exists. Is the inverse function convex?
5. Let f be a continuous function on (a, b) satisfying
   $$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(f(x) + f(y)\Big), \quad \forall x, y \in (a, b).$$
   Show that f is convex. Suggestion: Show
   $$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n}\ ,$$
   for $n = 2^m$.
6. Let f be differentiable on $[a, b]$. Show that it is convex if and only if
   $$f(y) - f(x) \ge f'(x)(y - x), \qquad \forall x, y \in [a, b].$$
   What is the geometric meaning of this inequality?
7. Establish the following two inequalities
   (a)
   $$\sin x + \sin y + \sin z \le \frac{3\sqrt{3}}{2}\ .$$
   (b)
   $$\sin x\ \sin y\ \sin z \le \frac{3\sqrt{3}}{8}\ .$$
   (c)
   $$\frac{1}{3}\left(\frac{1}{\sin x} + \frac{1}{\sin y} + \frac{1}{\sin z}\right) \ge \frac{2}{\sqrt{3}}\ .$$
   Here x, y, z are the three interior angles of a triangle.